Iterated integer tiles and self-similar spectral measures

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Overview

Backgrounds

Definition

Let μ be a Borel probability measure on $\mathbb{R}^d$ with compact support, μ is called a spectral measure if there is a countable set $\Lambda \subset \mathbb{R}^d$ such that

$$E(\Lambda) = \left\{ e^{-2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda \right\}$$

forms an orthonormal basis for $L^2(\mu)$.

- Spectral set Ω: $L^2(\Omega)$ admits an orthonormal basis $E(\Lambda)$.
- Question: What measures μ are spectral?

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- Question: What measures μ are spectral?

- (He, Lai and Lau, 2013)

  (Law of pure type) A spectral measure must be purely discrete, purely absolutely continuous or purely singularly continuous w.r.t . Lebesgue measure.

  Discrete case:

  If $\mu = \sum_{d \in D} p_d \delta_d$ is spectral, then D must be finite and $p_d = \frac{1}{\#D}$

- (Dutkay and Lai, 2015)

  If $d\mu = f(x)dx$ is spectral, then $f(x) = \mathbf{1}_{\Omega}(x)$ a.e.

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- Fuglede's conjecture: Ω is a spectral set if and only if Ω is a translational tile.

  Tao generalized the Fuglede's conjecture to the finite abelian group $\mathbb{Z}_n^d$.

  Based on the existence of a rational spectrum, the Fuglede's conjecture for $\mathbb{R}$ and $\mathbb{Z}_n$ are equivalent.
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The first singular spectral measure

Theorem (Jorgensen-Pedersen, 1998)

The $\frac{1}{4}$*−Cantor measures* $\mu_4 := \mu_{4,\{0,2\}}$

$$\mu_4(E) = \frac{1}{2}\mu_4(4E) + \frac{1}{2}\mu_4(4E-2)$$

is a spectral measure. The $\frac{1}{3}$*−Cantor measures* $\mu_3 := \mu_{3,\{0,2\}}$

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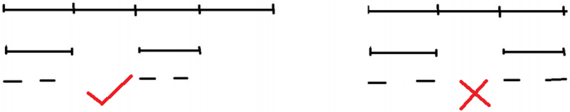

- (Hu-Lau, 2008; Dai, 2012)
  Bernoulli measure $\mu_{\rho,\{0,2\}}$ is spectral if and only if $\rho^{-1} = 2q$ is an even integer.

- From the measure-equation

$$\mu_{4,\{0,2\}}(E) = \frac{1}{2}\mu_{4,\{0,2\}}(4E) + \frac{1}{2}\mu_{4,\{0,2\}}(4E-2),$$

we have

$$\begin{aligned}\mu_{4,\{0,2\}} &= \delta_{4^{-1}\{0,2\}} * \mu_4(4\cdot) \\ &= \delta_{4^{-1}\{0,2\}} * \delta_{4^{-2}\{0,2\}} * \delta_{4^{-3}\{0,2\}} * \cdots\end{aligned}$$

where $\delta_D = \frac{1}{\#D}\sum_{d\in D}\delta_d$. So

$$\mu_{4,\{0,2\}} * \mu_{4,\{0,1\}} = \mu_{4,\{0,1,2,3\}} = \mathcal{L}_{[0,1]}.$$

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Generalized Fuglede's Conjecture

Conjecture (Gabardo-Lai, 2014)

A compactly supported Borel probability measure μ on $\mathbb{R}^d$ is spectral if and only if there exists a Borel probability ν on $\mathbb{R}^d$ and a fundamental domain Q of some lattice on $\mathbb{R}^d$ such that $\mu * \nu = \mathcal{L}_Q$.

- The generalized Fuglede's Conjecture implies the classical Fuglede's Conjecture on$\mathbb{R}$.
- (A. and He; Lai and Gabadord): If $\mu * \nu = \mathcal{L}_{[0,1]}$, then both of μ and ν are spectral.

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- (A. and He; Lai and Gabadord): If $\mu * \nu = \mathcal{L}_{[0,1]}$, then both of μ and ν are spectral.

- Key point: If $\mu * \nu = \mathcal{L}_{[0,1]}$, then they are random convolutions

$$\mu = \delta_{\frac{1}{b_1} D_1} * \cdots * \delta_{\frac{1}{b_1 b_2 \cdots b_{2k+1}} D_{2k+1}} * \cdots$$

$$\nu = \delta_{\frac{1}{b_1 b_2} D_2} * \cdots * \delta_{\frac{1}{b_1 b_2 \cdots b_{2k}} D_{2k}} * \cdots ,$$

where $D_k = \{0, 1, \cdots, N_k - 1\}$ and $N_k | b_k$.

- The spectrality of random convolutions was first studied by Strichart (2000).
- If all $(b_k, D_k) \equiv (b, D)$, then they are self-similar measure.

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Self-similar measures

- Let $b > 1$ be an integer, $D \subset \mathbb{Z}$ be a finite set with cardinality N,

  a self-similar set

$$K(b, D) = \sum_{k=1}^{\infty} b^{-k} D.$$

  a self-similar measure is

$$\mu_{b,D} = \delta_{b^{-1}D} * \delta_{b^{-2}D} * \delta_{b^{-3}D} * \cdots$$

- $[0, 1]$ is also a self-similar set as

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- (Deng and Chen, 2022): For any integer-pair (b, D), if $\mu_{b,D}$ is a spectral measure, then the OSC holds.

- (Shief) If the OSC holds, then $\dim_H = \frac{\ln N}{\ln b}$ and

$$0 < \mathcal{H}^{\frac{\ln N}{\ln b}}\Big(K(b, D)\Big) < \infty,$$

- If $\mu_{b,D}$ is a spectral measure, then it is the nomalized $\frac{\ln N}{\ln b}$-Hausdorff measure supported on $K(b, D)$,

$$\mu_{b,D} = \frac{1}{\mathcal{H}^{\frac{\ln N}{\ln b}}(K(b, D))}\mathcal{H}^{\frac{\ln N}{\ln b}}|_{K(b,D)}.$$

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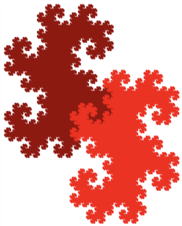
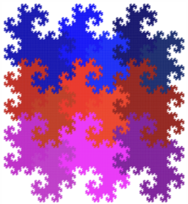
Fuglede's conjecture for self-similar measures

- Suppose $\#D = b$, $K(b, D)$ is called a self-similar tile if it has positive Lebesgue measure.
- (Lagarias and Wang, 1996): A self-similar tile must be a translational tile
- Twin dragon tile

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$





Self-similar tile on $\mathbb{R}$

(Lagarias and Wang, 1996): Suppose $0 \in D \subset \mathbb{R}$ with $\#D = b$

- $K(b, D)$ is a self-similar tile $\quad \Rightarrow \quad \alpha\mathcal{D} \subset \mathbb{Z}$ for some $\alpha \neq 0$.
- $K(b, D)$ is a self-similar tile with $0 \in D \subset \mathbb{Z}$
  $\Leftrightarrow$ the sum $D_{b,k} := D + bD + \cdots + b^k D$ is direct for each $k \geq 1$;
  $\Rightarrow$ each iteration $D_{b,k} = D \oplus bD \oplus \cdots \oplus b^k D$ is an integer tile.

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Spectral measure ⇒ Tile

Theorem

Suppose $D \subset \mathbb{Z}$ with $\#\mathcal{D} = b$, if $\mu_{b,D}$ is a spectral measure, then $K(b, D)$ is a self-similar tile.

Proof Sketch:

- Suppose $0 \in D \subset \mathbb{Z}$ with $\#\mathcal{D} = b$ and $\gcd(D) = 1$.
- $\mu_{b,D}$ is a spectral measure

  ⇒ OSC (Deng and Chen, 2022)

  ⇔ the sum $D_{b,k} = D + bD + \cdots + b^{k-1}D$ is direct for each $k \geq 1$; (He and Lau, 2008)

  ⇔ $K(b, D)$ is a self-similar tile (Lagarias and Wang, 1996).

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Tile $\Rightarrow$ Spectral measure

- (Fu, He and Lau, 2015): strict product-form digit

Theorem (A. and Lai, 2023; Li and Rao, 2025+)

Suppose C-M's conjecture is true. If $K(b, D)$ is a self-similar tile, then it is a spectral set.

Question: For the singularly continuous case ($\#D < b$), how about the spectral measures-tiling connection?

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- (Łaba and Wang, 2002):

  Let D be a complementing set $\pmod b$ with $b \geq 1$. Suppose that $\#D$ has no more than two distinct prime factors. Then $\mu_{b,D}$ is a spectral measure

- Key point 1: D is a complementing set $\pmod b$, it means D tiles $\mathbb{Z}_b$, i.e., there exists a $C \subset \mathbb{Z}$ such that

$$D \oplus C \equiv \{0, 1, \cdots, b-1\} \pmod b.$$

  In this case, $K(b, D \oplus C)$ is a fundamental domain of $\mathbb{Z}$ and of course it is a tile. Then

$$\mu_{b,D} * \mu_{b,C} = \mathcal{L}_{K(b,D\oplus C)}.$$

- Key point 2: $\#D$ has no more than two distinct prime factors. In this case, the Coven-Meyerowitz conjecture is true: tiles D of finite group $\mathbb{Z}_b$ is spectral (Łaba).

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- We say $K(b, D)$ weakly tiles a self-similar set if there is a $C \subset \mathbb{Z}$ such that $K(b, D \oplus C)$ forms a self-similar tile.

Theorem (A. and He, preprint)

Suppose that the C-M conjecture is true and $K(b, D)$ weakly tiles a self-similar set, then $\mu_{b,D}$ is a spectral measure.

- For which pair (b, D), $K(b, D)$ can weakly tile a self-similar set?
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- We say $K(b, D)$ weakly tiles a self-similar set if there is a $C \subset \mathbb{Z}$ such that $K(b, D \oplus C)$ forms a self-similar tile.

Theorem (A. and He, preprint)

Suppose that the C-M conjecture is true and $K(b, D)$ weakly tiles a self-similar set, then $\mu_{b,D}$ is a spectral measure.

- For which pair (b, D), $K(b, D)$ can weakly tile a self-similar set?
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- Question: If D tiles $\mathbb{Z}$ (or $\mathbb{Z}_{b^n}$), can $K(b,D)$ weakly tile a self-similar set?

  Answer: : Might NOT!

  (1) $D = \{0,1,32,33\} = \{0,1\} + 16\{0,2\}$ tiles $\mathbb{Z}_{16^2}$, YES.

  (2) $\widetilde{D} = \{0,1,16,17\} = \{0,1\} + 16\{0,1\}$ tiles $\mathbb{Z}_{16^2}$, NO;

- Denote $D_{b,k} = D + bD + \cdots + b^{k-1}D$, then

$$K(b,D) = \sum_{k=1}^{\infty} b^{-k} D = K(b^k, D_{b,k}).$$

- each $D \oplus 16D$ tiles $\mathbb{Z}$, and so is $D \oplus \cdots \oplus 16^{k-1}D$ for each $k \geq 1$.

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- Suppose $D \subset \mathbb{Z}$, we call D an iterated integer tile with respect to b if for each $k \geq 1$, the sum

$$D_{b,k} := D + bD + \cdots + b^{k-1}D$$

is direct and tiles $\mathbb{Z}$.

Theorem (A. and He, preprint)

$T(b, D)$ weakly tiles a self-similar set if and only if D is an iterated integer tile with respect to b

Some remarks:

- If D tiles $\mathbb{Z}_b$, then it is an iterated integer tile with respect to b.
- D is an integer tile and the sum in $D_{b,k}$ is direct $\not\Rightarrow$ each $D_{b,k}$ is an integer tile e.g. $b = 3, D = \{0, 1\}$ ($\frac{1}{3}$−Cantor measure $\mu_{b,\{0,1\}}$).

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Spectral $\Rightarrow$ Tile

- (Łaba and Wang, 2002) conjecture that:
  For any integer-pair (b, D), if $\mu_{b,D}$ is spectral, then D is an integer tile.
- Recall that

$$\begin{aligned} \mu_{b,D} &= \delta_{b^{-1}D} * \delta_{b^{-2}D} * \delta_{b^{-3}D} * \cdots \\ &= \delta_{b^{-k}(D+bD+\cdots+b^{k-1}D)} * \delta_{b^{-2k}(D+bD+\cdots+b^{k-1}D)} \\ &= \mu_{b^k, D_{b,k}} \end{aligned}$$

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- Suppose $0 \in \Lambda$ is a spectrum of $\mu_{b,D}$
- The orthogonality implies that $\widehat{\mu}_{b,D}(\lambda) = 0$ for any nonzero elements $\lambda \in \Lambda$, that is to say

$$\Lambda \setminus \{0\} \subset \mathcal{Z}(\widehat{\mu}_{b,D}) = \bigcup_{k=1}^{\infty} b^k \mathcal{Z}(\widehat{\delta_D})$$

where $\mathcal{Z}(f) := \{\xi : f(\xi) = 0\}$.

Theorem (A. and He, 2025+)

Suppose $D \subset \mathbb{Z}$ with $\mathcal{Z}(\widehat{\delta_D}) \subset \frac{1}{pq}(\mathbb{Z} \setminus \mathbb{Z})$ where p, q are primes, then the following are equivalent: (1)$\mu_{b,D}$ is a spectral measure;

(2) D is an iterated integer tile;

(3) $K(b, D)$ can weakly tile a self-similar set.

(the generalized Fuglede's conjecture is true for this measure.)

Structure of the iterated integer tile

Theorem (A. and He, preprint)

$T(b,D)$ weakly tiles a self-similar set if and only if D is an iterated integer tile with respect to b

Question: for which pair (b,D), is $T(b,D)$ a self-similar tile?

The known results

- $b=p$, p is a prime number (Bandt, 1991; Kenyon, 1992)
- $b=p^{k+1}$, p is a prime number (Lagarias and Wang, 1996)
- $b=pq$, p,q are different prime numbers (Lau and Rao, 2003)
- $b=p^{\alpha}q$, p,q are different prime numbers (Lai, Lau and Rao, 2017)

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- (Bandt, 1991): If D is a complete residue set mod b and $\gcd(D) = 1$, then $K(b, D)$ is a fundamental domain of $\mathbb{Z}$;
- (Odlyzko, 1978; Lagarias and Wang, 1996)
  Strict product-form: Given $\mathcal{E} = \mathcal{E}_0 \oplus \cdots \oplus \mathcal{E}_k = \{0, 1, \cdots, b-1\}$,

$$D = \mathcal{E}_0 \oplus b^{\ell_1}\mathcal{E}_1 \oplus \cdots \oplus b^{\ell_k}\mathcal{E}_k, \quad 0 \le \ell_1 \le \cdots \le \ell_k.$$

  In this case,

$$\begin{aligned} T(b, D) &= \sum_{j=1}^{\infty} b^{-j}\left(\mathcal{E}_0 \oplus b^{\ell_1}\mathcal{E}_1 \oplus \cdots \oplus b^{\ell_k}\mathcal{E}_k\right) \\ &= T(b, \mathcal{E}) \oplus \mathcal{A} \\ &= [0, 1] + \mathcal{A}. \end{aligned}$$

Theorem (A. and Lau, 2019)

Suppose $A = pI_2, p$ *is prime and* $D \subset \mathbb{Z}^2$ *with* (A, D) *is primitive.* $K(A, D)$ *is a self-similar tile if and only if*

$$D = \bigcup_{d \in D_0} \left(d + A^k B_d \right),$$

where $k \geq 1$ *and* $D_0 \oplus B_d$ *is a complete residue set modulus* A *for every* $d \in D_0$*.*

- (Li and Rao, 2025+) named it as skew-product-form digit set with respect to A

Suppose $0 \in D \subset \mathbb{Z}$ with $\#D = b$ and $\gcd(D) = 1$. Then $T(b, D)$ is a self-similar tile if and only if there is a $k \geq 1$ such that $D_{b,k}$ is a skew-product-form digit set with respect to b.

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Suppose $0 \in D \subset \mathbb{Z}$ with $\#D = b$ and $\gcd(D) = 1$. Then $T(b, D)$ is a self-similar tile if and only if there is a $k \geq 1$ such that $D_{b,k}$ is a skew-product-form digit set with respect to b.

the case $\#D \leq b$

- Quasi-product-form digit set with respect to b if

$$D = \bigcup_{a \in A} (a + bB_a),$$

and $A \oplus B_a$ tiles $\mathbb{Z}_b$.

Theorem (A. and He, preprint)

Suppose $0 \in D \subset \mathbb{Z}$ with $\#D \leq b$ and $\gcd(D) = 1$. Then D is an iterated integer tile w.r.t b if and only if there is a $k \geq 1$ and $\alpha > 0$ such that $\alpha \cdot D_{b,k}$ is a quasi-product-form digit set.

Thank you for your attention!